

16. Jacobian Elliptic Functions and Theta Functions

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16. Jacobian Elliptic Functions and Theta Functions

Mathematical Properties

Jacobian Elliptic Functions

16.1. Introduction

A doubly periodic meromorphic function is called an *elliptic function*.

Let m, m_1 be numbers such that

$$m + m_1 = 1.$$

We call m the *parameter*, m_1 the *complementary parameter*.

In what follows we shall assume that the parameter m is a real number. Without loss of generality we can then suppose that $0 \leq m \leq 1$ (see 16.10, 16.11).

We define *quarter-periods* K and iK' by

16.1.1

$$K(m) = K = \int_0^{\pi/2} \frac{d\theta}{(1 - m \sin^2 \theta)^{1/2}},$$

$$iK'(m) = iK' = i \int_0^{\pi/2} \frac{d\theta}{(1 - m_1 \sin^2 \theta)^{1/2}}$$

so that K and K' are real numbers. K is called the real, iK' the imaginary quarter-period.

We note that

$$16.1.2 \quad K(m) = K'(m_1) = K'(1 - m).$$

We also note that if any *one* of the numbers $m, m_1, K(m), K'(m), K'(m)/K(m)$ is given, all the rest are determined. Thus K and K' can not both be chosen arbitrarily.

In the Argand diagram denote the points $0, K, K + iK', iK'$ by s, c, d, n respectively. These points are at the vertices of a rectangle. The translations of this rectangle by $\lambda K, \mu iK'$, where λ, μ are given all integral values positive or negative, will lead to the lattice

s	c	s	c
n	d	n	d
s	c	s	c
n	d	n	d

the pattern being repeated indefinitely on all sides.

Let p, q be any two of the letters s, c, d, n . Then p, q determine in the lattice a minimum rectangle whose sides are of length K and K' and whose vertices s, c, d, n are in counterclockwise order.

Definition

The Jacobian elliptic function $pq u$ is defined by the following three properties.

(i) $pq u$ has a simple zero at p and a simple pole at q .

(ii) The step from p to q is a half-period of $pq u$. Those of the numbers $K, iK', K + iK'$ which differ from this step are only quarter-periods.

(iii) The coefficient of the leading term in the expansion of $pq u$ in ascending powers of u about $u=0$ is unity. With regard to (iii) the leading term is $u, 1/u, 1$ according as $u=0$ is a zero, a pole, or an ordinary point.

Thus the functions with a pole or zero at the origin (i.e., the functions in which one letter is s) are odd, and the others are even.

Should we wish to call explicit attention to the value of the parameter, we write $pq(u|m)$ instead of $pq u$.

The Jacobian elliptic functions can also be defined with respect to certain integrals. Thus if

$$16.1.3 \quad u = \int_0^\varphi \frac{d\theta}{(1 - m \sin^2 \theta)^{1/2}},$$

the angle φ is called the *amplitude*

$$16.1.4 \quad \varphi = \text{am } u$$

and we define

16.1.5

$$\text{sn } u = \sin \varphi, \quad \text{cn } u = \cos \varphi,$$

$$\text{dn } u = (1 - m \sin^2 \varphi)^{1/2} = \Delta(\varphi).$$

Similarly all the functions $pq u$ can be expressed in terms of φ . This second set of definitions, although seemingly different, is mathematically equivalent to the definition previously given in terms of a lattice. For further explanation of notations, including the interpretation, of such expressions as $\text{sn}(\varphi|\alpha), \text{cn}(u|m), \text{dn}(u, k)$, see 17.2.

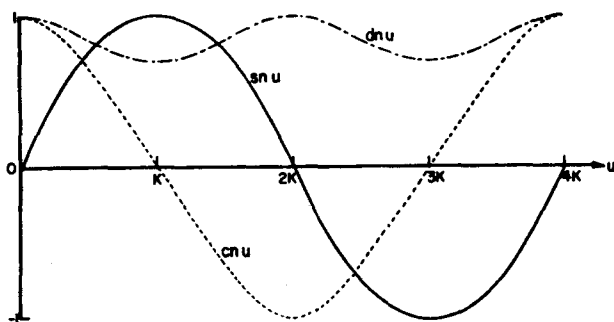
16.2. Classification of the Twelve Jacobian Elliptic Functions

According to Poles and Half-Periods

	Pole iK'	Pole $K+iK'$	Pole K	Pole 0	
Half period iK'	$\text{sn } u$	$\text{cd } u$	$\text{dc } u$	$\text{ns } u$	Periods $2iK'$, $4K+4iK'$, $4K$
Half period $K+iK'$	$\text{cn } u$	$\text{sd } u$	$\text{nc } u$	$\text{ds } u$	Periods $4iK'$, $2K+2iK'$, $4K$
Half period K	$\text{dn } u$	$\text{nd } u$	$\text{sc } u$	$\text{cs } u$	Periods $4iK'$, $4K+4iK'$, $2K$

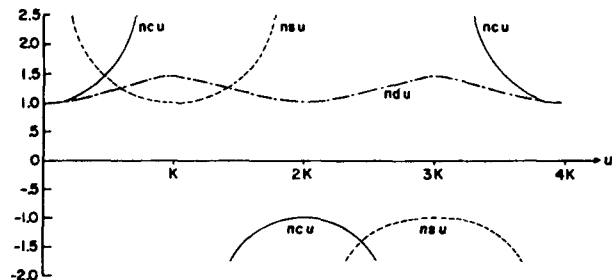
The three functions in a vertical column are *copolar*.

The four functions in a horizontal line are *coperiodic*. Of the periods quoted in the last line of each row only two are independent.

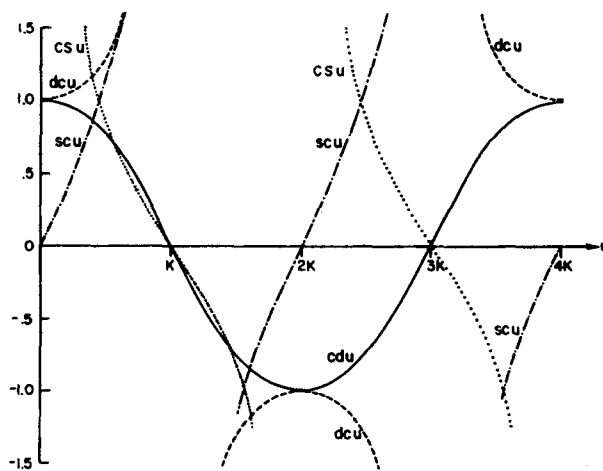
FIGURE 16.1. *Jacobian elliptic functions* $\text{sn } u, \text{cn } u, \text{dn } u$

$$m = \frac{1}{2}$$

The curve for $\text{cn}(u\frac{1}{2})$ is the boundary between those which have an inflexion and those which have not.

FIGURE 16.2. *Jacobian elliptic functions* $\text{ns } u, \text{nc } u, \text{nd } u$

$$m = \frac{1}{2}$$

FIGURE 16.3. *Jacobian elliptic functions* $\text{sc } u, \text{cs } u, \text{cd } u, \text{dc } u$

$$m = \frac{1}{2}$$

16.3. Relation of the Jacobian Functions to the Copolar Trio $\text{sn } u, \text{cn } u, \text{dn } u$

$$16.3.1 \quad \text{cd } u = \frac{\text{cn } u}{\text{dn } u} \quad \text{dc } u = \frac{\text{dn } u}{\text{cn } u} \quad \text{ns } u = \frac{1}{\text{sn } u}$$

$$16.3.2 \quad \text{sd } u = \frac{\text{sn } u}{\text{dn } u} \quad \text{nc } u = \frac{1}{\text{cn } u} \quad \text{ds } u = \frac{\text{dn } u}{\text{sn } u}$$

$$16.3.3 \quad \text{nd } u = \frac{1}{\text{dn } u} \quad \text{sc } u = \frac{\text{sn } u}{\text{cn } u} \quad \text{cs } u = \frac{\text{cn } u}{\text{sn } u}$$

And generally if p, q, r are any three of the letters s, c, d, n ,

$$16.3.4 \quad pq u = \frac{pr u}{qr u}$$

provided that when two letters are the same, e.g., $pp u$, the corresponding function is put equal to unity.

16.4. Calculation of the Jacobian Functions by Use of the Arithmetic-Geometric Mean (A.G.M.)

For the A.G.M. scale see 17.6.

To calculate $\text{sn } (u|m)$, $\text{cn } (u|m)$, and $\text{dn } (u|m)$ form the A.G.M. scale starting with

$$16.4.1 \quad a_0=1, b_0=\sqrt{m_1}, c_0=\sqrt{m},$$

terminating at the step N when c_N is negligible to the accuracy required. Find φ_N in degrees where

$$16.4.2 \quad \varphi_N = 2^N a_N u \frac{180^\circ}{\pi}$$

and then compute successively $\varphi_{N-1}, \varphi_{N-2}, \dots$, φ_1, φ_0 from the recurrence relation

$$16.4.3 \quad \sin (2\varphi_{n-1} - \varphi_n) = \frac{c_n}{a_n} \sin \varphi_n.$$

Then

16.4.4

$$\text{sn } (u|m) = \sin \varphi_0, \text{ cn } (u|m) = \cos \varphi_0$$

$$\text{dn } (u|m) = \frac{\cos \varphi_0}{\cos (\varphi_1 - \varphi_0)}.$$

From these all the other functions can be determined.

16.5. Special Arguments

	u	$\text{sn } u$	$\text{cn } u$	$\text{dn } u$
16.5.1	0	0	1	1
16.5.2	$\frac{1}{2}K$	$\frac{1}{(1+m_1^{1/2})^{1/2}}$	$\frac{m_1^{1/4}}{(1+m_1^{1/2})^{1/2}}$	$m_1^{1/4}$
16.5.3	K	1	0	$m_1^{1/2}$
16.5.4	$\frac{1}{2}(iK')$	$im^{-1/4}$	$\frac{(1+m^{1/2})^{1/2}}{m^{1/4}}$	$(1+m^{1/2})^{1/2}$
* 16.5.5	$\frac{1}{2}(K+iK')$	$2^{-1/2}m^{-1/4}[(1+m^{1/2})^{1/2} + i(1-m^{1/2})^{1/2}]$	$\left(\frac{m_1}{4m}\right)^{1/4}(1-i)$	$\left(\frac{m_1}{4}\right)^{1/4}[(1+m_1^{1/2})^{1/2} - i(1-m_1^{1/2})^{1/2}]$
16.5.6	$K + \frac{1}{2}(iK')$	$m^{-1/4}$	$-i\left(\frac{1-m^{1/2}}{m^{1/2}}\right)^{1/2}$	$(1-m^{1/2})^{1/2}$
16.5.7	iK'	∞	∞	∞
16.5.8	$\frac{1}{2}K + iK'$	$(1-m_1^{1/2})^{-1/2}$	$-i\left(\frac{m_1^{1/2}}{1-m_1^{1/2}}\right)^{1/2}$	$-im_1^{1/4}$
16.5.9	$K + iK'$	$m^{-1/2}$	$-i(m_1/m)^{1/2}$	0

16.6. Jacobian Functions when $m=0$ or 1

		$m=0$	$m=1$
16.6.1	$\text{sn } (u m)$	$\sin u$	$\tanh u$
16.6.2	$\text{cn } (u m)$	$\cos u$	$\text{sech } u$
16.6.3	$\text{dn } (u m)$	1	$\text{sech } u$
16.6.4	$\text{cd } (u m)$	$\cos u$	1
16.6.5	$\text{sd } (u m)$	$\sin u$	$\sinh u$
16.6.6	$\text{nd } (u m)$	1	$\cosh u$
16.6.7	$\text{dc } (u m)$	$\sec u$	1
16.6.8	$\text{nc } (u m)$	$\sec u$	$\cosh u$
16.6.9	$\text{sc } (u m)$	$\tan u$	$\sinh u$
16.6.10	$\text{ns } (u m)$	$\csc u$	$\coth u$
16.6.11	$\text{ds } (u m)$	$\csc u$	$\text{csch } u$
16.6.12	$\text{cs } (u m)$	$\cot u$	$\text{csch } u$
16.6.13	$\text{am } (u m)$	u	$\text{gd } u$

16.7. Principal Terms

When the elliptic function $pq\ u$ is expanded in ascending powers of $(u-K_r)$, where K_r is one of $0, K, iK', K+iK'$, the first term of the expansion is called the principal term and has one of the forms $A, B \times (u-K_r), C \div (u-K_r)$ according as K_r is an ordinary point, a zero, or a pole of $pq\ u$. The following list gives these forms, where \times means that the factor $(u-K_r)$ has to be supplied and \div means that the divisor $(u-K_r)$ has to be supplied.

	$K_r =$	0	K	iK'	$K+iK'$
16.7.1	sn u	$1 \times$	1	$m^{-1/2} \div$	$m^{-1/2}$
16.7.2	cn u	1	$-m_1^{1/2} \times$	$-im^{-1/2} \div$	$-i \left(\frac{m_1}{m}\right)^{1/2}$
16.7.3	dn u	1	$m_1^{1/2}$	$-i \div$	$im_1^{1/2} \times$
16.7.4	cd u	1	$-1 \times$	$m^{-1/2}$	$-m^{-1/2} \div$
16.7.5	sd u	$1 \times$	$m_1^{-1/2}$	$im^{-1/2}$	$-\frac{1}{(mm_1)^{1/2}} \div$
16.7.6	nd u	1	$m_1^{-1/2}$	$i \times$	$-im_1^{-1/2} \div$
16.7.7	dc u	1	$-1 \div$	$m^{1/2}$	$-m^{1/2} \times$
16.7.8	nc u	1	$-m_1^{-1/2} \div$	$im^{1/2} \times$	$i \left(\frac{m}{m_1}\right)^{1/2}$
16.7.9	sc u	$1 \times$	$-m_1^{-1/2} \div$	i	$im_1^{-1/2}$
16.7.10	ns u	$1 \div$	1	$m^{1/2} \times$	$m^{1/2}$
16.7.11	ds u	$1 \div$	$m_1^{1/2}$	$-im^{1/2}$	$i(mm_1)^{1/2} \times$
16.7.12	cs u	$1 \div$	$-m_1^{1/2} \times$	$-i$	$-im_1^{1/2}$

16.8. Change of Argument

		u	$-u$	$u+K$	$u-K$	$K-u$	$u+2K$	$u-2K$	$2K-u$	$u+iK'$	$u+2iK'$	$u+K+iK'$	$u+2K+2iK'$
16.8.1	sn	sn u	$-\text{sn } u$	cd u	$-\text{cd } u$	cd u	$-\text{sn } u$	$-\text{sn } u$	sn u	$m^{-1/2}\text{ns } u$	sn u	$m^{-1/2}\text{dc } u$	$-\text{sn } u$
16.8.2	cn	cn u	cn u	$-m_1^{1/2}\text{sd } u$	$m_1^{1/2}\text{sd } u$	$m_1^{1/2}\text{sd } u$	$-\text{cn } u$	$-\text{cn } u$	$-\text{cn } u$	$-im^{-1/2}\text{ds } u$	$-\text{cn } u$	$-im_1^{1/2}m^{-1/2}\text{nc } u$	cn u
16.8.3	dn	dn u	dn u	$m_1^{1/2}\text{nd } u$	$m_1^{1/2}\text{nd } u$	$m_1^{1/2}\text{nd } u$	dn u	dn u	dn u	$-\text{ics } u$	$-\text{dn } u$	$im_1^{1/2}\text{sc } u$	$-\text{dn } u$
16.8.4	cd	cd u	cd u	$-\text{sn } u$	sn u	sn u	$-\text{cd } u$	$-\text{cd } u$	$-\text{cd } u$	$m^{-1/2}\text{dc } u$	cd u	$-\text{sn } u$	$-\text{cd } u$
16.8.5	sd	sd u	$-\text{sd } u$	$m_1^{-1/2}\text{cn } u$	$-m_1^{-1/2}\text{cn } u$	$m_1^{-1/2}\text{cn } u$	$-\text{sd } u$	$-\text{sd } u$	sd u	$im^{-1/2}\text{nc } u$	$-\text{sd } u$	$-im_1^{-1/2}m^{-1/2}\text{ds } u$	sd u
16.8.6	nd	nd u	nd u	$m_1^{-1/2}\text{dn } u$	$m_1^{-1/2}\text{dn } u$	$m_1^{-1/2}\text{dn } u$	nd u	nd u	nd u	$\text{isc } u$	$-\text{nd } u$	$-im_1^{-1/2}\text{cs } u$	$-\text{nd } u$
16.8.7	dc	dc u	dc u	$-\text{ns } u$	ns u	ns u	$-\text{dc } u$	$-\text{dc } u$	$-\text{dc } u$	$m^{1/2}\text{cd } u$	dc u	$-\text{sn } u$	$-\text{dc } u$
16.8.8	nc	nc u	nc u	$-m_1^{-1/2}\text{ds } u$	$m_1^{-1/2}\text{ds } u$	$m_1^{-1/2}\text{ds } u$	$-\text{nc } u$	$-\text{nc } u$	$-\text{nc } u$	$im^{1/2}\text{sd } u$	$-\text{nc } u$	$im_1^{-1/2}m^{1/2}\text{cn } u$	nc u
16.8.9	sc	sc u	$-\text{sc } u$	$-m_1^{-1/2}\text{cs } u$	$-m_1^{-1/2}\text{cs } u$	$m_1^{-1/2}\text{cs } u$	sc u	sc u	sc u	$\text{ind } u$	sc u	$im_1^{-1/2}\text{dn } u$	$-\text{sc } u$
16.8.10	ns	ns u	$-\text{ns } u$	dc u	$-\text{dc } u$	dc u	$-\text{ns } u$	$-\text{ns } u$	ns u	$m^{1/2}\text{sn } u$	ns u	$m^{1/2}\text{cd } u$	$-\text{ns } u$
16.8.11	ds	ds u	$-\text{ds } u$	$m_1^{1/2}\text{nc } u$	$-m_1^{1/2}\text{nc } u$	$m_1^{1/2}\text{nc } u$	$-\text{ds } u$	$-\text{ds } u$	ds u	$-im^{1/2}\text{cn } u$	$-\text{ds } u$	$im_1^{1/2}m^{1/2}\text{sd } u$	ds u
16.8.12	cs	cs u	$-\text{cs } u$	$-m_1^{1/2}\text{sc } u$	$-m_1^{1/2}\text{sc } u$	$m_1^{1/2}\text{sc } u$	cs u	cs u	$-\text{cs } u$	$-\text{idn } u$	$-\text{cs } u$	$-im_1^{1/2}\text{nd } u$	$-\text{cs } u$

16.9. Relations Between the Squares of the Functions

$$16.9.1 \quad -\operatorname{dn}^2 u + m_1 = -m \operatorname{cn}^2 u = m \operatorname{sn}^2 u - m$$

$$16.9.2 \quad -m_1 \operatorname{nd}^2 u + m_1 = -m m_1 \operatorname{sd}^2 u = m \operatorname{cd}^2 u - m$$

$$16.9.3 \quad m_1 \operatorname{sc}^2 u + m_1 = m_1 \operatorname{nc}^2 u = \operatorname{dc}^2 u - m$$

$$16.9.4 \quad \operatorname{cs}^2 u + m_1 = \operatorname{ds}^2 u = \operatorname{ns}^2 u - m$$

In using the above results remember that $m + m_1 = 1$.

If $pq u$, $rt u$ are any two of the twelve functions, one entry expresses $tq^2 u$ in terms of $pq^2 u$ and another expresses $qt^2 u$ in terms of $rt^2 u$. Since $tq^2 u \cdot qt^2 u = 1$, we can obtain from the table the bilinear relation between $pq^2 u$ and $rt^2 u$. Thus for the functions $\operatorname{cd} u$, $\operatorname{sn} u$ we have

$$16.9.5 \quad \operatorname{nd}^2 u = \frac{1 - m \operatorname{cd}^2 u}{m_1}, \quad \operatorname{dn}^2 u = 1 - m \operatorname{sn}^2 u$$

and therefore

$$16.9.6 \quad (1 - m \operatorname{cd}^2 u)(1 - m \operatorname{sn}^2 u) = m_1.$$

16.10. Change of Parameter

Negative Parameter

If m is a positive number, let

$$16.10.1 \quad \mu = \frac{m}{1+m}, \quad \mu_1 = \frac{1}{1+m}, \quad v = \frac{u}{\mu_1^{1/2}} \quad (0 < \mu < 1)$$

$$16.10.2 \quad \operatorname{sn}(u|-m) = \mu_1^{1/2} \operatorname{sd}(v|\mu)$$

$$16.10.3 \quad \operatorname{cn}(u|-m) = \operatorname{cd}(v|\mu)$$

$$16.10.4 \quad \operatorname{dn}(u|-m) = \operatorname{nd}(v|\mu).$$

16.11. Reciprocal Parameter (Jacobi's Real Transformation)

$$16.11.1 \quad m > 0, \quad \mu = m^{-1}, \quad v = um^{1/2}$$

$$16.11.2 \quad \operatorname{sn}(u|m) = \mu^{1/2} \operatorname{sn}(v|\mu)$$

$$16.11.3 \quad \operatorname{cn}(u|m) = \operatorname{dn}(v|\mu)$$

$$16.11.4 \quad \operatorname{dn}(u|m) = \operatorname{cn}(v|\mu)$$

Here if $m > 1$ then $m^{-1} = \mu < 1$.

Thus elliptic functions whose parameter is real can be made to depend on elliptic functions whose parameter lies between 0 and 1.

16.12. Descending Landen Transformation (Gauss' Transformation)

To decrease the parameter, let

$$16.12.1 \quad \mu = \left(\frac{1 - m_1^{1/2}}{1 + m_1^{1/2}} \right)^2, \quad v = \frac{u}{1 + \mu^{1/2}},$$

then

$$16.12.2 \quad \operatorname{sn}(u|m) = \frac{(1 + \mu^{1/2}) \operatorname{sn}(v|\mu)}{1 + \mu^{1/2} \operatorname{sn}^2(v|\mu)}$$

$$16.12.3 \quad \operatorname{cn}(u|m) = \frac{\operatorname{cn}(v|\mu) \operatorname{dn}(v|\mu)}{1 + \mu^{1/2} \operatorname{sn}^2(v|\mu)}$$

$$16.12.4 \quad \operatorname{dn}(u|m) = \frac{\operatorname{dn}^2(v|\mu) - (1 - \mu^{1/2})}{(1 + \mu^{1/2}) - \operatorname{dn}^2(v|\mu)}.$$

Note that successive applications can be made conveniently to find $\operatorname{sn}(u|m)$ in terms of $\operatorname{sn}(v|\mu)$ and $\operatorname{dn}(u|m)$ in terms of $\operatorname{dn}(v|\mu)$, but that the calculation of $\operatorname{cn}(u|m)$ requires all three functions.

16.13. Approximation in Terms of Circular Functions

When the parameter m is so small that we may neglect m^2 and higher powers, we have the approximations

16.13.1

$$\operatorname{sn}(u|m) \approx \sin u - \frac{1}{4} m(u - \sin u \cos u) \cos u$$

16.13.2

$$\operatorname{cn}(u|m) \approx \cos u + \frac{1}{4} m(u - \sin u \cos u) \sin u$$

$$16.13.3 \quad \operatorname{dn}(u|m) \approx 1 - \frac{1}{2} m \sin^2 u$$

$$16.13.4 \quad \operatorname{am}(u|m) \approx u - \frac{1}{4} m(u - \sin u \cos u).$$

One way of calculating the Jacobian functions is to use Landen's descending transformation to reduce the parameter sufficiently for the above formulae to become applicable. See also 16.14.

16.14. Ascending Landen Transformation

To increase the parameter, let

$$16.14.1 \quad \mu = \frac{4m^{1/2}}{(1+m^{1/2})^2}, \quad \mu_1 = \left(\frac{1-m^{1/2}}{1+m^{1/2}} \right)^2, \quad v = \frac{u}{1+\mu_1^{1/2}}$$

$$16.14.2 \quad \operatorname{sn}(u|m) = (1 + \mu_1^{1/2}) \frac{\operatorname{sn}(v|\mu) \operatorname{cn}(v|\mu)}{\operatorname{dn}(v|\mu)}$$

$$16.14.3 \quad \operatorname{cn}(u|m) = \frac{1 + \mu_1^{1/2}}{\mu} \frac{\operatorname{dn}^2(v|\mu) - \mu_1^{1/2}}{\operatorname{dn}(v|\mu)}$$

$$16.14.4 \quad \operatorname{dn}(u|m) = \frac{1 - \mu_1^{1/2}}{\mu} \frac{\operatorname{dn}^2(v|\mu) + \mu_1^{1/2}}{\operatorname{dn}(v|\mu)}$$

Note that, when successive applications are to be made, it is simplest to calculate $\text{dn}(u|m)$ since this is expressed always in terms of the same function. The calculation of $\text{cn}(u|m)$ leads to that of $\text{dn}(v|\mu)$.

The calculation of $\text{sn}(u|m)$ necessitates the evaluation of all three functions.

16.15. Approximation in Terms of Hyperbolic Functions

When the parameter m is so close to unity that m_1^2 and higher powers of m_1 can be neglected we have the approximations

16.15.1

$$\text{sn}(u|m) \approx \tanh u + \frac{1}{4} m_1 (\sinh u \cosh u - u) \text{sech}^2 u$$

16.15.2

$$\text{cn}(u|m) \approx \text{sech } u$$

$$-\frac{1}{4} m_1 (\sinh u \cosh u - u) \tanh u \text{sech } u$$

16.15.3

$$\text{dn}(u|m) \approx \text{sech } u$$

$$+\frac{1}{4} m_1 (\sinh u \cosh u + u) \tanh u \text{sech } u$$

16.15.4

$$\text{am}(u|m) \approx \text{gd } u + \frac{1}{4} m_1 (\sinh u \cosh u - u) \text{sech } u.$$

Another way of calculating the Jacobian functions is to use Landen's ascending transformation to increase the parameter sufficiently for the above formulae to become applicable. See also 16.13.

16.16. Derivatives

	Function	Derivative
16.16.1 16.16.2 16.16.3	$\text{sn } u$ $\text{cn } u$ $\text{dn } u$	$\text{cn } u \text{ dn } u$ $-\text{sn } u \text{ dn } u$ $-m \text{ sn } u \text{ cn } u$ Pole n
16.16.4 16.16.5 16.16.6	$\text{cd } u$ $\text{sd } u$ $\text{nd } u$	$-m_1 \text{ sd } u \text{ nd } u$ $\text{cd } u \text{ nd } u$ $m \text{ sd } u \text{ cd } u$ Pole d
16.16.7 16.16.8 16.16.9	$\text{dc } u$ $\text{nc } u$ $\text{sc } u$	$m_1 \text{ sc } u \text{ nc } u$ $\text{sc } u \text{ dc } u$ $\text{dc } u \text{ nc } u$ Pole c
16.16.10 16.16.11 16.16.12	$\text{ns } u$ $\text{ds } u$ $\text{cs } u$	$-\text{ds } u \text{ cs } u$ $-\text{cs } u \text{ ns } u$ $-\text{ns } u \text{ ds } u$ Pole s

Note that the derivative is proportional to the product of the two copolar functions.

16.17. Addition Theorems

16.17.1 $\text{sn}(u+v)$

$$= \frac{\text{sn } u \cdot \text{cn } v \cdot \text{dn } v + \text{sn } v \cdot \text{cn } u \cdot \text{dn } u}{1 - m \text{ sn}^2 u \cdot \text{sn}^2 v}$$

16.17.2 $\text{cn}(u+v)$

$$= \frac{\text{cn } u \cdot \text{cn } v - \text{sn } u \cdot \text{dn } u \cdot \text{sn } v \cdot \text{dn } v}{1 - m \text{ sn}^2 u \cdot \text{sn}^2 v}$$

16.17.3 $\text{dn}(u+v)$

$$= \frac{\text{dn } u \cdot \text{dn } v - m \text{ sn } u \cdot \text{cn } u \cdot \text{sn } v \cdot \text{cn } v}{1 - m \text{ sn}^2 u \cdot \text{sn}^2 v}$$

Addition theorems are derivable one from another and are expressible in a great variety of forms. Thus $\text{ns}(u+v)$ comes from $1/\text{sn}(u+v)$ in the form $(1 - m \text{ sn}^2 u \text{ sn}^2 v) / (\text{sn } u \text{ cn } v \text{ dn } v + \text{sn } v \text{ cn } u \text{ dn } u)$ from 16.17.1.

Alternatively $\text{ns}(u+v) = m^{1/2} \text{sn} \{ (iK' - u) - v \}$ which again from 16.17.1 yields the form $(\text{ns } u \text{ cs } v \text{ ds } u - \text{ns } v \text{ cs } u \text{ ds } v) / (\text{sn}^2 u - \text{sn}^2 v)$.

The function $\text{pq}(u+v)$ is a rational function of the four functions $\text{pq } u$, $\text{pq } v$, $\text{pq}' u$, $\text{pq}' v$.

16.18. Double Arguments

16.18.1 $\text{sn } 2u$

$$= \frac{2 \text{sn } u \cdot \text{cn } u \cdot \text{dn } u}{1 - m \text{ sn}^4 u} = \frac{2 \text{sn } u \cdot \text{cn } u \cdot \text{dn } u}{\text{cn}^2 u + \text{sn}^2 u \cdot \text{dn}^2 u}$$

16.18.2 $\text{cn } 2u$

$$= \frac{\text{cn}^2 u - \text{sn}^2 u \cdot \text{dn}^2 u}{1 - m \text{ sn}^4 u} = \frac{\text{cn}^2 u - \text{sn}^2 u \cdot \text{dn}^2 u}{\text{cn}^2 u + \text{sn}^2 u \cdot \text{dn}^2 u}$$

16.18.3 $\text{dn } 2u$

$$= \frac{\text{dn}^2 u - m \text{ sn}^2 u \cdot \text{cn}^2 u}{1 - m \text{ sn}^4 u} = \frac{\text{dn}^2 u + \text{cn}^2 u (\text{dn}^2 u - 1)}{\text{dn}^2 u - \text{cn}^2 u (\text{dn}^2 u - 1)}$$

16.18.4

$$\frac{1 - \text{cn } 2u}{1 + \text{cn } 2u} = \frac{\text{sn}^2 u \cdot \text{dn}^2 u}{\text{cn}^2 u}$$

16.18.5

$$\frac{1 - \text{dn } 2u}{1 + \text{dn } 2u} = \frac{m \text{ sn}^2 u \cdot \text{cn}^2 u}{\text{dn}^2 u}$$

16.19. Half Arguments

16.19.1

$$\text{sn}^2 \frac{1}{2} u = \frac{1 - \text{cn } u}{1 + \text{dn } u}$$

16.19.2

$$\text{cn}^2 \frac{1}{2} u = \frac{\text{dn } u + \text{cn } u}{1 + \text{dn } u}$$

16.19.3

$$\text{dn}^2 \frac{1}{2} u = \frac{m_1 + \text{dn } u + m \text{ cn } u}{1 + \text{dn } u}$$

16.20. Jacobi's Imaginary Transformation

16.20.1

$$\text{sn}(iu|m) = i \text{sc}(u|m_1)$$

16.20.2

$$\text{cn}(iu|m) = \text{nc}(u|m_1)$$

16.20.3

$$\text{dn}(iu|m) = \text{dc}(u|m_1)$$

16.21. Complex Arguments

With the abbreviations

16.21.1

$$s = \operatorname{sn}(x|m), c = \operatorname{cn}(x|m), d = \operatorname{dn}(x|m), s_1 = \operatorname{sn}(y|m_1), \\ c_1 = \operatorname{cn}(y|m_1), d_1 = \operatorname{dn}(y|m_1)$$

$$16.21.2 \quad \operatorname{sn}(x+iy|m) = \frac{s \cdot d_1 + ic \cdot d \cdot s_1 \cdot c_1}{c_1^2 + ms^2 \cdot s_1^2}$$

$$16.21.3 \quad \operatorname{cn}(x+iy|m) = \frac{c \cdot c_1 - is \cdot d \cdot s_1 \cdot d_1}{c_1^2 + ms^2 \cdot s_1^2}$$

$$16.21.4 \quad \operatorname{dn}(x+iy|m) = \frac{d \cdot c_1 \cdot d_1 - ims \cdot c \cdot s_1}{c_1^2 + ms^2 \cdot s_1^2}$$

16.22. Leading Terms of the Series in Ascending Powers of u

16.22.1

$$\operatorname{sn}(u|m) = u - (1+m) \frac{u^3}{3!} + (1+14m+m^2) \frac{u^5}{5!} \\ - (1+135m+135m^2+m^3) \frac{u^7}{7!} + \dots$$

16.22.2

$$\operatorname{cn}(u|m) = 1 - \frac{u^2}{2!} + (1+4m) \frac{u^4}{4!} \\ - (1+44m+16m^2) \frac{u^6}{6!} + \dots$$

16.22.3

$$\operatorname{dn}(u|m) = 1 - m \frac{u^2}{2!} + m(4+m) \frac{u^4}{4!} \\ - m(16+44m+m^2) \frac{u^6}{6!} + \dots$$

No formulae are known for the general coefficients in these series.

16.23. Series Expansions in Terms of the Nome $q = e^{-\pi K'/K}$ and the Argument $v = \pi u/(2K)$

$$16.23.1 \quad \operatorname{sn}(u|m) = \frac{2\pi}{m^{1/2}K} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1-q^{2n+1}} \sin(2n+1)v$$

$$16.23.2 \quad \operatorname{cn}(u|m) = \frac{2\pi}{m^{1/2}K} \sum_{n=0}^{\infty} \frac{q^{n+1/2}}{1+q^{2n+1}} \cos(2n+1)v$$

$$16.23.3 \quad \operatorname{dn}(u|m) = \frac{\pi}{2K} + \frac{2\pi}{K} \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \cos 2nv$$

16.23.4

$$\operatorname{cd}(u|m) = \frac{2\pi}{m^{1/2}K} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+1/2}}{1-q^{2n+1}} \cos(2n+1)v$$

16.23.5

$$\operatorname{sd}(u|m) = \frac{2\pi}{(mm_1)^{1/2}K} \sum_{n=0}^{\infty} (-1)^n \frac{q^{n+1/2}}{1+q^{2n+1}} \sin(2n+1)v$$

16.23.6

$$\operatorname{nd}(u|m) = \frac{\pi}{2m_1^{1/2}K} + \frac{2\pi}{m_1^{1/2}K} \sum_{n=1}^{\infty} (-1)^n \frac{q^n}{1+q^{2n}} \cos 2nv$$

16.23.7

$$\operatorname{dc}(u|m) = \frac{\pi}{2K} \sec v \\ + \frac{2\pi}{K} \sum_{n=0}^{\infty} (-1)^n \frac{q^{2n+1}}{1-q^{2n+1}} \cos(2n+1)v$$

16.23.8

$$\operatorname{nc}(u|m) = \frac{\pi}{2m_1^{1/2}K} \sec v \\ - \frac{2\pi}{m_1^{1/2}K} \sum_{n=0}^{\infty} (-1)^n \frac{q^{2n+1}}{1+q^{2n+1}} \cos(2n+1)v$$

16.23.9

$$\operatorname{sc}(u|m) = \frac{\pi}{2m_1^{1/2}K} \tan v \\ + \frac{2\pi}{m_1^{1/2}K} \sum_{n=1}^{\infty} (-1)^n \frac{q^{2n}}{1+q^{2n}} \sin 2nv$$

16.23.10

$$\operatorname{ns}(u|m) = \frac{\pi}{2K} \csc v - \frac{2\pi}{K} \sum_{n=0}^{\infty} \frac{q^{2n+1}}{1-q^{2n+1}} \sin(2n+1)v$$

16.23.11

$$\operatorname{ds}(u|m) = \frac{\pi}{2K} \csc v - \frac{2\pi}{K} \sum_{n=0}^{\infty} \frac{q^{2n+1}}{1+q^{2n+1}} \sin(2n+1)v$$

16.23.12

$$\operatorname{cs}(u|m) = \frac{\pi}{2K} \cot v - \frac{2\pi}{K} \sum_{n=1}^{\infty} \frac{q^{2n}}{1+q^{2n}} \sin 2nv$$

16.24. Integrals of the Twelve Jacobian Elliptic Functions

$$16.24.1 \quad \int \operatorname{sn} u \, du = m^{-1/2} \ln(\operatorname{dn} u - m^{1/2} \operatorname{cn} u)$$

$$16.24.2 \quad \int \operatorname{cn} u \, du = m^{-1/2} \arccos(\operatorname{dn} u)$$

$$16.24.3 \quad \int \operatorname{dn} u \, du = \arcsin(\operatorname{sn} u)$$

$$16.24.4 \quad \int \operatorname{cd} u \, du = m^{-1/2} \ln(\operatorname{nd} u + m^{1/2} \operatorname{sd} u)$$

$$16.24.5 \quad \int \operatorname{sd} u \, du = (mm_1)^{-1/2} \arcsin(-m^{1/2} \operatorname{cd} u)$$

$$16.24.6 \quad \int \operatorname{nd} u \, du = m_1^{-1/2} \arccos(\operatorname{cd} u)$$

$$16.24.7 \quad \int \operatorname{dc} u \, du = \ln(\operatorname{nc} u + \operatorname{sc} u)$$

$$16.24.8 \quad \int \operatorname{nc} u \, du = m_1^{-1/2} \ln(\operatorname{dc} u + m_1^{1/2} \operatorname{sc} u)$$

$$16.24.9 \quad \int \operatorname{sc} u \, du = m_1^{-1/2} \ln(\operatorname{dc} u + m_1^{1/2} \operatorname{nc} u)$$

$$16.24.10 \quad \int \operatorname{ns} u \, du = \ln(\operatorname{ds} u - \operatorname{cs} u)$$

$$16.24.11 \quad \int \operatorname{ds} u \, du = \ln(\operatorname{ns} u - \operatorname{cs} u)$$

$$16.24.12 \quad \int \operatorname{cs} u \, du = \ln(\operatorname{ns} u - \operatorname{ds} u)$$

In numerical use of the above table certain restrictions must be put on u in order to keep the arguments of the logarithms positive and to avoid

trouble with many-valued inverse circular functions.

16.25. Notation for the Integrals of the Squares of the Twelve Jacobian Elliptic Functions

$$16.25.1 \quad \text{Pq } u = \int_0^u \text{pq}^2 t \, dt \text{ when } q \neq s$$

$$16.25.2 \quad \text{Ps } u = \int_0^u \left(\text{pq}^2 t - \frac{1}{t^2} \right) dt - \frac{1}{u}$$

Examples

$$\text{Cd } u = \int_0^u \text{cd}^2 t \, dt, \text{Ns } u = \int_0^u \left(\text{ns}^2 t - \frac{1}{t^2} \right) dt - \frac{1}{u}$$

16.26. Integrals in Terms of the Elliptic Integral of the Second Kind (see 17.4)

$$16.26.1 \quad m\text{Sn } u = -E(u) + u$$

$$16.26.2 \quad m\text{Cn } u = E(u) - m_1 u \quad \text{Pole } n$$

$$16.26.3 \quad \text{Dn } u = E(u)$$

$$16.26.4 \quad m\text{Cd } u = -E(u) + u + m\text{sn } u \, \text{cd } u$$

$$16.26.5 \quad m\text{m}_1\text{Sd } u = E(u) - m_1 u - m\text{sn } u \, \text{cd } u \quad \text{Pole } d$$

$$16.26.6 \quad m_1\text{Nd } u = E(u) - m\text{sn } u \, \text{cd } u$$

$$16.26.7 \quad \text{Dc } u = -E(u) + u + \text{sn } u \, \text{dc } u$$

$$16.26.8 \quad m_1\text{Nc } u = -E(u) + m_1 u + \text{sn } u \, \text{dc } u \quad \text{Pole } c$$

$$16.26.9 \quad m_1\text{Sc } u = -E(u) + \text{sn } u \, \text{dc } u$$

$$16.26.10 \quad \text{Ns } u = -E(u) + u - \text{cn } u \, \text{ds } u$$

$$16.26.11 \quad \text{Ds } u = -E(u) + m_1 u - \text{cn } u \, \text{ds } u \quad \text{Pole } s$$

$$16.26.12 \quad \text{Cs } u = -E(u) - \text{cn } u \, \text{ds } u$$

All the above may be expressed in terms of Jacobi's zeta function (see 17.4.27).

$$Z(u) = E(u) - \frac{E}{K} u, \text{ where } E = E(K)$$

16.27. Theta Functions; Expansions in Terms of the Nome q

$$16.27.1 \quad \vartheta_1(z, q) = \vartheta_1(z) = 2q^{1/4} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \sin(2n+1)z$$

$$16.27.2 \quad \vartheta_2(z, q) = \vartheta_2(z) = 2q^{1/4} \sum_{n=0}^{\infty} q^{n(n+1)} \cos(2n+1)z$$

$$16.27.3 \quad \vartheta_3(z, q) = \vartheta_3(z) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos 2nz$$

$$16.27.4 \quad \vartheta_4(z, q) = \vartheta_4(z) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nz$$

Theta functions are important because every one of the Jacobian elliptic functions can be expressed as the ratio of two theta functions. See 16.36.

The notation shows these functions as depending on the variable z and the nome q , $|q| < 1$. In this case, here and elsewhere, the convergence is not dependent on the trigonometrical terms. In their relation to the Jacobian elliptic functions, we note that the nome q is given by

$$q = e^{-\pi K'/K},$$

where K and iK' are the quarter periods. Since $q = q(m)$ is determined when the parameter m is given, we can also regard the theta functions as dependent upon m and then we write

$$\vartheta_a(z, q) = \vartheta_a(z|m), \quad a = 1, 2, 3, 4$$

but when no ambiguity is to be feared, we write $\vartheta_a(z)$ simply.

The above notations are those given in Modern Analysis [16.6].

There is a bewildering variety of notations, for example the function $\vartheta_4(z)$ above is sometimes denoted by $\vartheta_0(z)$ or $\vartheta(z)$; see the table given in Modern Analysis [16.6]. Further the argument $u = 2Kz/\pi$ is frequently used so that in consulting books caution should be exercised.

16.28. Relations Between the Squares of the Theta Functions

$$16.28.1 \quad \vartheta_1^2(z) \vartheta_1^2(0) = \vartheta_3^2(z) \vartheta_3^2(0) - \vartheta_2^2(z) \vartheta_2^2(0)$$

$$16.28.2 \quad \vartheta_2^2(z) \vartheta_4^2(0) = \vartheta_1^2(z) \vartheta_2^2(0) - \vartheta_3^2(z) \vartheta_3^2(0)$$

$$16.28.3 \quad \vartheta_3^2(z) \vartheta_4^2(0) = \vartheta_4^2(z) \vartheta_3^2(0) - \vartheta_1^2(z) \vartheta_2^2(0)$$

$$16.28.4 \quad \vartheta_4^2(z) \vartheta_1^2(0) = \vartheta_3^2(z) \vartheta_3^2(0) - \vartheta_2^2(z) \vartheta_2^2(0)$$

$$16.28.5 \quad \vartheta_2^4(0) + \vartheta_4^4(0) = \vartheta_3^4(0)$$

Note also the important relation

$$16.28.6 \quad \vartheta_1'(0) = \vartheta_2(0) \vartheta_3(0) \vartheta_4(0) \text{ or } \vartheta_1' = \vartheta_2 \vartheta_3 \vartheta_4$$

16.29. Logarithmic Derivatives of the Theta Functions

$$16.29.1 \quad \frac{\vartheta_1'(u)}{\vartheta_1(u)} = \cot u + 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{1 - q^{2n}} \sin 2nu$$

16.29.2

$$\frac{\vartheta_2'(u)}{\vartheta_2(u)} = -\tan u + 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^{2n}}{1-q^{2n}} \sin 2nu$$

$$16.29.3 \quad \frac{\vartheta_3'(u)}{\vartheta_3(u)} = 4 \sum_{n=1}^{\infty} (-1)^n \frac{q^n}{1-q^{2n}} \sin 2nu$$

$$16.29.4 \quad \frac{\vartheta_4'(u)}{\vartheta_4(u)} = 4 \sum_{n=1}^{\infty} \frac{q^n}{1-q^{2n}} \sin 2nu$$

16.30. Logarithms of Theta Functions of Sum and Difference

16.30.1

$$\ln \frac{\vartheta_1(\alpha+\beta)}{\vartheta_1(\alpha-\beta)} = \ln \frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} + 4 \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^{2n}}{1-q^{2n}} \sin 2n\alpha \sin 2n\beta$$

16.30.2

$$\ln \frac{\vartheta_2(\alpha+\beta)}{\vartheta_2(\alpha-\beta)} = \ln \frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{q^{2n}}{1-q^{2n}} \sin 2n\alpha \sin 2n\beta$$

16.30.3

$$\ln \frac{\vartheta_3(\alpha+\beta)}{\vartheta_3(\alpha-\beta)} = 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{q^n}{1-q^{2n}} \sin 2n\alpha \sin 2n\beta$$

16.30.4

$$\ln \frac{\vartheta_4(\alpha+\beta)}{\vartheta_4(\alpha-\beta)} = 4 \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1-q^{2n}} \sin 2n\alpha \sin 2n\beta$$

The corresponding expressions when $\beta = i\gamma$ are easily deduced by use of the formulae 4.3.55 and 4.3.56.

16.31. Jacobi's Notation for Theta Functions

$$16.31.1 \quad \Theta(u|m) = \Theta(u) = \vartheta_4(v), \quad v = \frac{\pi u}{2K}$$

$$16.31.2 \quad \Theta_1(u|m) = \Theta_1(u) = \vartheta_3(v) = \Theta(u+K)$$

$$16.31.3 \quad H(u|m) = H(u) = \vartheta_1(v)$$

$$16.31.4 \quad H_1(u|m) = H_1(u) = \vartheta_2(v) = H(u+K)$$

16.32. Calculation of Jacobi's Theta Function $\Theta(u|m)$ by Use of the Arithmetic-Geometric Mean

Form the A.G.M. scale starting with

$$16.32.1 \quad a_0 = 1, b_0 = \sqrt{m_1}, c_0 = \sqrt{m}$$

terminating with the N th step when c_N is negligible to the accuracy required. Find φ_N in degrees, where

$$16.32.2 \quad \varphi_N = 2^N a_N u \frac{180^\circ}{\pi}$$

and then compute successively $\varphi_{N-1}, \varphi_{N-2}, \dots, \varphi_1, \varphi_0$ from the recurrence relation

$$16.32.3 \quad \sin(2\varphi_{n-1} - \varphi_n) = \frac{c_n}{a_n} \sin \varphi_n.$$

Then

16.32.4

$$\begin{aligned} \ln \Theta(u|m) = & \frac{1}{2} \ln \frac{2m_1^{1/2} K(m)}{\pi} + \frac{1}{2} \ln \frac{\cos(\varphi_1 - \varphi_0)}{\cos \varphi_0} \\ & + \frac{1}{4} \ln \sec(2\varphi_0 - \varphi_1) + \frac{1}{8} \ln \sec(2\varphi_1 - \varphi_2) + \dots \\ & + \frac{1}{2^{N+1}} \ln \sec(2\varphi_{N-1} - \varphi_N) \end{aligned}$$

16.33. Addition of Quarter-Periods to Jacobi's Eta and Theta Functions

u	$-u$	$u+K$	$u+2K$	$u+iK'$	$u+2iK'$	$u+K+iK'$	$u+2K+2iK'$
16.33.1 $H(u)$	$-H(u)$	$H_1(u)$	$-H(u)$	$iM(u)\Theta(u)$	$-N(u)H(u)$	$M(u)\Theta_1(u)$	$N(u)H(u)$
16.33.2 $H_1(u)$	$H_1(u)$	$-H(u)$	$-H_1(u)$	$M(u)\Theta_1(u)$	$N(u)H_1(u)$	$-iM(u)\Theta(u)$	$-N(u)H_1(u)$
16.33.3 $\Theta_1(u)$	$\Theta_1(u)$	$\Theta(u)$	$\Theta_1(u)$	$M(u)H_1(u)$	$N(u)\Theta_1(u)$	$iM(u)H(u)$	$N(u)\Theta_1(u)$
16.33.4 $\Theta(u)$	$\Theta(u)$	$\Theta_1(u)$	$\Theta(u)$	$iM(u)H(u)$	$-N(u)\Theta(u)$	$M(u)H_1(u)$	$-N(u)\Theta(u)$

where

$$M(u) = \left[\exp\left(-\frac{\pi i u}{2K}\right) \right] q^{-1/4},$$

$$N(u) = \left[\exp\left(-\frac{\pi i u}{K}\right) \right] q^{-1/4}$$

$H(u)$ and $H_1(u)$ have the period $4K$. $\Theta(u)$ and $\Theta_1(u)$ have the period $2K$.

$2iK'$ is a quasi-period for all four functions, that is to say, increase of the argument by $2iK'$ multiplies the function by a factor.

16.34. Relation of Jacobi's Zeta Function to the Theta Functions

$$Z(u) = \frac{\partial}{\partial u} \ln \Theta(u)$$

$$16.34.1 \quad Z(u) = \frac{\pi}{2K} \frac{\vartheta'_1\left(\frac{\pi u}{2K}\right)}{\vartheta_1\left(\frac{\pi u}{2K}\right)} - \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u}$$

$$16.34.2 \quad = \frac{\pi}{2K} \frac{\vartheta'_2\left(\frac{\pi u}{2K}\right)}{\vartheta_2\left(\frac{\pi u}{2K}\right)} + \frac{\operatorname{dn} u \operatorname{sn} u}{\operatorname{cn} u}$$

$$16.34.3 \quad = \frac{\pi}{2K} \frac{\vartheta'_3\left(\frac{\pi u}{2K}\right)}{\vartheta_3\left(\frac{\pi u}{2K}\right)} - m \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}$$

$$16.34.4 \quad = \frac{\pi}{2K} \frac{\vartheta'_4\left(\frac{\pi u}{2K}\right)}{\vartheta_4\left(\frac{\pi u}{2K}\right)}$$

16.35. Calculation of Jacobi's Zeta Function $Z(u|m)$ by Use of the Arithmetic-Geometric Mean

Form the A.G.M. scale 17.6 starting with

$$16.35.1 \quad a_0 = 1, b_0 = \sqrt{m_1}, c_0 = \sqrt{m}$$

terminating at the N th step when c_N is negligible to the accuracy required. Find φ_N in degrees where

$$16.35.2 \quad \varphi_N = 2^N a_N u \frac{180^\circ}{\pi}$$

and then compute successively $\varphi_{N-1}, \varphi_{N-2}, \dots, \varphi_1, \varphi_0$ from the recurrence relation

$$16.35.3 \quad \sin(2\varphi_{N-1} - \varphi_N) = \frac{c_N}{a_N} \sin \varphi_N.$$

Then

16.35.4

$$Z(u|m) = c_1 \sin \varphi_1 + c_2 \sin \varphi_2 + \dots + c_N \sin \varphi_N.$$

16.36. Neville's Notation for Theta Functions

These functions are defined in terms of Jacobi's theta functions of 16.31 by

$$16.36.1 \quad \vartheta_s(u) = \frac{H(u)}{H'(0)}, \vartheta_c(u) = \frac{H(u+K)}{H(K)}$$

$$16.36.2 \quad \vartheta_d(u) = \frac{\Theta(u+K)}{\Theta(K)}, \vartheta_n(u) = \frac{\Theta(u)}{\Theta(0)}.$$

If λ, μ are any integers positive, negative, or zero the points $u_0 + 2\lambda K + 2\mu iK'$ are said to be *congruent* to u_0 .

$\vartheta_s(u)$ has zeros at the points congruent to 0
 $\vartheta_c(u)$ has zeros at the points congruent to K
 $\vartheta_n(u)$ has zeros at the points congruent to iK'
 $\vartheta_d(u)$ has zeros at the points congruent to $K + iK'$

Thus the suffix secures that the function $\vartheta_p(u)$ has zeros at the points marked p in the introductory diagram in 16.1.2, and the constant by which Jacobi's function is divided secures that the leading coefficient of $\vartheta_p(u)$ at the origin is unity. Therefore the functions have the fundamentally important property that if p, q are any two of the letters s, c, n, d , the Jacobian elliptic function $pq u$ is given by

$$16.36.3 \quad pq u = \frac{\vartheta_p(u)}{\vartheta_q(u)}.$$

These functions also have the property

$$16.36.4 \quad m_1^{-1/4} \vartheta_c(K-u) = \vartheta_s(u)$$

$$16.36.5 \quad m_1^{-1/4} \vartheta_d(K-u) = \vartheta_n(u),$$

for complementary arguments u and $K-u$.

In terms of the theta functions defined in 16.27, let $v = \pi u/(2K)$, then

$$16.36.6 \quad \vartheta_s(u) = \frac{2K\vartheta_1(v)}{\vartheta_1'(0)}, \vartheta_c(u) = \frac{\vartheta_2(v)}{\vartheta_2(0)}$$

$$16.36.7 \quad \vartheta_d(u) = \frac{\vartheta_3(v)}{\vartheta_3(0)}, \vartheta_n(u) = \frac{\vartheta_4(v)}{\vartheta_4(0)}.$$

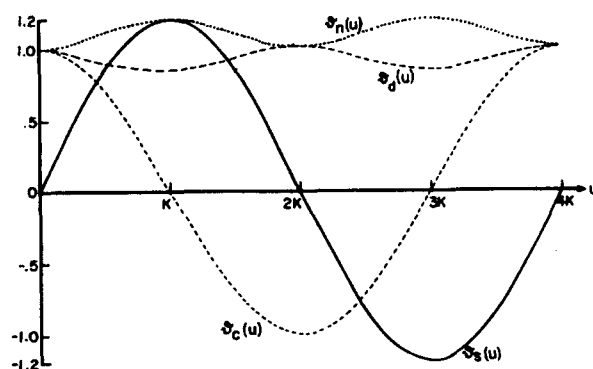


FIGURE 16.4. Neville's theta functions
 $\vartheta_s(u), \vartheta_c(u), \vartheta_d(u), \vartheta_n(u)$
 $m = \frac{1}{2}$

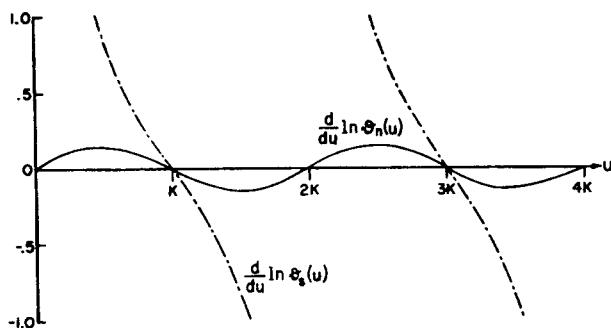


FIGURE 16.5. Logarithmic derivatives of theta functions

$$\frac{d}{du} \ln \vartheta_2(u), \frac{d}{du} \ln \vartheta_3(u)$$

$$m = \frac{1}{2}$$

16.37. Expression as Infinite Products

$$q = q(m), v = \pi u / (2K)$$

16.37.1

$$\vartheta_2(u) = \left(\frac{16q}{m m_1} \right)^{1/6} \sin v \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2v + q^{4n})$$

16.37.2

$$\vartheta_3(u) = \left(\frac{16q m_1^{1/2}}{m} \right)^{1/6} \cos v \prod_{n=1}^{\infty} (1 + 2q^{2n} \cos 2v + q^{4n})$$

16.37.3

$$\vartheta_4(u) = \left(\frac{m m_1}{16q} \right)^{1/12} \prod_{n=1}^{\infty} (1 + 2q^{2n-1} \cos 2v + q^{4n-2})$$

16.37.4

$$\vartheta_5(u) = \left(\frac{m}{16q m_1^2} \right)^{1/12} \prod_{n=1}^{\infty} (1 - 2q^{2n-1} \cos 2v + q^{4n-2})$$

Numerical Methods**16.39. Use and Extension of the Tables**

Example 1. Calculate $nc(1.99650|.64)$ to 4S.

From Table 17.1, $1.99650 = K + .001$. From the table of principal terms

$$nc u = -m_1^{-1/2} / (u - K) + \dots$$

$$nc(K + .001|.64) = \frac{-(.36)^{-1/2}}{.001} + \dots$$

$$= -\frac{10000}{6} + \dots$$

$$= -1667 + \dots$$

and since the next term is of order .001 this value -1667 is correct to at least 4S.

Example 2. Use the descending Landen transformation to calculate $dn(.20|.19)$ to 6D.

Here $m = .19$, $m_1^{1/2} = .9$ and so from 16.12.1

$$\mu = \left(\frac{1}{19} \right)^2, 1 + \mu^{1/2} = \frac{20}{19}, v = .19.$$

Also

16.38. Expression as Infinite Series

$$\text{Let } v = \pi u / (2K)$$

16.38.1

$$\vartheta_2(u) = \left[\frac{2\pi q^{1/2}}{m^{1/2} m_1^{1/2} K} \right]^{1/2} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \sin(2n+1)v$$

$$16.38.2 \quad \vartheta_3(u) = \left[\frac{2\pi q^{1/2}}{m^{1/2} K} \right]^{1/2} \sum_{n=0}^{\infty} q^{n(n+1)} \cos(2n+1)v$$

$$16.38.3 \quad \vartheta_4(u) = \left[\frac{\pi}{2K} \right]^{1/2} \left\{ 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos 2nv \right\}$$

16.38.4

$$\vartheta_5(u) = \left[\frac{\pi}{2m_1^{1/2} K} \right]^{1/2} \left\{ 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nv \right\}$$

$$16.38.5 \quad (2K/\pi)^{1/2} = 1 + 2q + 2q^4 + 2q^9 + \dots = \vartheta_3(0, q)$$

16.38.6

$$(2K'/\pi)^{1/2} = 1 + 2q_1 + 2q_1^4 + 2q_1^9 + \dots = \vartheta_3(0, q_1)$$

16.38.7

$$(2m^{1/2} K/\pi)^{1/2} = 2q^{1/4} (1 + q^2 + q^6 + q^{12} + q^{20} + \dots) = \vartheta_2(0, q)$$

16.38.8

$$(2m_1^{1/2} K/\pi)^{1/2} = 1 - 2q + 2q^4 - 2q^9 + \dots = \vartheta_4(0, q).$$

$$\mu^2 = \left(\frac{1}{19} \right)^4 = 10^{-8} \times 7.67$$

which is negligible.

From 16.12.4

$$dn(.20|.19) = \frac{dn \left[.19 \middle| \left(\frac{1}{19} \right)^2 \right] - \left(1 - \frac{1}{19} \right)}{\left(1 + \frac{1}{19} \right) - dn^2 \left[.19 \middle| \left(\frac{1}{19} \right)^2 \right]}$$

Now from 16.13.3

$$dn \left[.19 \middle| \left(\frac{1}{19} \right)^2 \right] = .999951$$

whence $dn(.20|.19) = .996253$.

Example 3. Use the ascending Landen transformation to calculate $dn(.20|.81)$ to 5D.

From 16.14.1

$$\mu = \frac{4(.9)}{(1.9)^2} = \frac{360}{361}, \mu_1 = \left(\frac{1}{19} \right)^2$$

$$1 + \mu_1^{1/2} = \frac{20}{19}, v = \frac{19}{20} \times .20 = .19,$$

μ_1^2 is negligible to 4D. Thus

$$\operatorname{dn}(.20|.81) = \frac{19}{20} \times \frac{\operatorname{dn}^2\left(.19 \left| \frac{360}{361} \right.\right) + \frac{1}{19}}{\operatorname{dn}\left(.19 \left| \frac{360}{361} \right.\right)}$$

From 16.15.3

$$\begin{aligned} \operatorname{dn}\left(.19 \left| \frac{360}{361} \right.\right) &= \operatorname{sech}(.19) + \frac{1}{4} \times \frac{1}{361} \tanh .19 \operatorname{sech} .19 \\ &\quad [\sinh .19 \cosh .19 + .19] \\ &= .982218 + \frac{1}{4} \times \frac{1}{361} (.187746)(.982218) \\ &\quad [(.191145)(1.01810) + .19] \\ &= .982218 + \frac{1}{4} \times \frac{1}{361} (.184408)[.384605] \\ &= .982218 + .000049 = .982267. \end{aligned}$$

Thus $\operatorname{dn}(.20|.81) = .98406$.

Example 4. Use the ascending Landen transformation to calculate $\operatorname{cn}(.20|.81)$ to 6D.

Using 16.14.4, we calculate $\operatorname{dn}(.20|.81)$ and deduce $\operatorname{cn}(.20|.81)$ from 16.14.3 settling the sign from Figure 16.1.

As in the preceding example, we reduce the calculation of $\operatorname{dn}(.20|.81)$ to that of $\operatorname{dn}\left(.19 \left| \frac{360}{361} \right.\right)$, when

$$\operatorname{dn}\left(.19 \left| \frac{360}{361} \right.\right) = .982267$$

$$\operatorname{dn}(.20|.81) = .984056$$

$$\operatorname{cn}(.20|.81) = .980278.$$

Example 5. Use the A.G.M. scale to compute $\operatorname{dc}(.672|.36)$ to 4D.

From 16.9.6 we have $\operatorname{dc}^2(.672|.36) = .36 + \frac{.64}{1 - \operatorname{sn}^2(.672|.36)}$. We now calculate $\operatorname{sn}(.672|.36)$ by the method given in 16.4. Form the A.G.M. scale

n	a_n	b_n	c_n	$\frac{c_n}{a_n}$	φ_n	$\sin \varphi_n$	$\sin (2\varphi_{n-1} - \varphi_n)$	$2\varphi_{n-1} - \varphi_n$
0	1	.8	.6	.6	.65546	.60952		
1	.9	.89443	.1	.11111	1.2069	.93452	.10383	.10402
2	.89721	.89721	.00279	.00311	2.4117	.66679	.00207	.00207
3	.89721	.89721	0	0	4.8234	-.99384	0	0

$$\varphi_n = 2^n a_n u \quad \varphi_3 = 2^3 (.89721)(.672) = 4.8234$$

continuing until $c_n = 0$ to 5D.

Then complete as indicated in 16.4 to find φ_0 and so $\operatorname{sn} u$ and hence $\operatorname{dc} u$,

$$\varphi_0 = .65546 \quad \operatorname{sn} u = .60952 \quad \operatorname{dc} u = 1.1740.$$

Example 6. Use the A.G.M. scale to compute $\Theta(.6|.36)$ to 5D.

We use the method explained in 16.32 with $a_0 = 1$, $b_0 = .8$, $c_0 = .6$.

Computing the A.G.M. as explained in 17.6, we find

(For values of a_n , b_n , c_n , see Example 5.)

n	φ_n	$\sin \varphi_n$	$\sin (2\varphi_{n-1} - \varphi_n)$	$2\varphi_{n-1} - \varphi_n$	$\sec (2\varphi_{n-1} - \varphi_n)$	$\frac{1}{2^{n+1}} \ln \sec (2\varphi_{n-1} - \varphi_n)$
0	.58803	.55472				
1	1.0780	.88101	.09789	.09805	1.0048	.00120
2	2.1533	.83509	.00260	.00260	1.	0
3	4.3066	-.91879	0	0	1.	0

and then complete the calculation outlined in 16.32 to give

$$\begin{aligned} \ln \Theta(u|m) &= -.05734 + .02935 + .00120 \\ &= -.02679 \\ \Theta(u|m) &= .97357. \end{aligned}$$

The series expansion for Θ is preferable.

Example 7. Use the q -series to compute $\text{cs}(.53601\ 62|.09)$.

Here we use the series 16.23.12, $K=1.60804\ 862$, $q=.00589\ 414$, $v=\frac{\pi u}{2K}=\frac{\pi}{6}$ radians or 30° .

Since q^4 is negligible to 8D, we have to 7D $\text{cs}(.53601\ 62|.09)$

$$\begin{aligned} &= \frac{\pi}{2K} \cot 30^\circ - \frac{2\pi}{K} \left\{ \frac{q^2}{1+q^2} \sin 60^\circ \right\} \\ &= (.97683\ 3852)(1.73205\ 081) \\ &\quad - 3.90733\ 541[(.00003\ 4740)(.86602\ 5404)] \\ &= 1.69180\ 83. \end{aligned}$$

Example 8. Use theta functions to compute $\text{sn}(.61802|.5)$ to 5D.

Here $K(\frac{1}{2})=1.85407$

$$\epsilon^\circ = \frac{.61802}{1.85407} \times 90^\circ = 30^\circ$$

$$\sin^2 \alpha = 1/2, \alpha = 45^\circ.$$

Thus

$$\begin{aligned} \text{sn}(.61802|.5) &= \frac{\vartheta_2(30^\circ \backslash 45^\circ)}{\vartheta_2(30^\circ \backslash 45^\circ)} \\ &= \frac{.59128}{1.04729} = .56458 \end{aligned}$$

from Table 16.1.

Example 9. Use theta functions to compute $\text{sc}(.61802|.5)$ to 5D.

As in the preceding example

$$\epsilon^\circ = 30^\circ, \alpha^\circ = 45^\circ$$

so that

$$\text{sc}(.61802|.5) = \frac{\vartheta_2(30^\circ \backslash 45^\circ)}{\vartheta_2(30^\circ \backslash 45^\circ)}$$

We use Table 16.1 to give

$$\vartheta_2(30^\circ \backslash 45^\circ) = .59128$$

$$(\sec 45^\circ)^{\frac{1}{2}} \vartheta_2(30^\circ \backslash 45^\circ) = 1.02796.$$

Therefore

$$\begin{aligned} \text{sc}(.61802|.5) &= \frac{.59128}{1.02796} (\sec 45^\circ)^{\frac{1}{2}} \\ &= .68402. \end{aligned}$$

Example 10. Find $\text{sn}(.75342|.7)$ by inverse interpolation in Table 17.5.

This method is explained in chapter 17, Example 7.

Example 11. Find u , given that $\text{cs}(u|.5) = .75$. From 16.9.4 we have

$$\text{sn}^2 u = \frac{1}{1 + \text{cs}^2 u}.$$

Thus

$$\text{sn}^2(u|.5) = .64$$

and

$$\text{sn}(u|.5) = .8.$$

We have therefore replaced the problem by that of finding u given $\text{sn}(u|m)$, where m is known. If $\varphi = \text{am } u$

$$\sin \varphi = \text{sn } u \text{ and so}$$

$$\varphi = .9272952 \text{ radians or } 53.13010^\circ.$$

From Table 17.5,

$$u = F(53.13010^\circ \backslash 45^\circ) = .99391.$$

Alternatively, starting with the above value of φ we can use the A.G.M. scale to calculate $F(\varphi \backslash \alpha)$ as explained in 17.6. This method is to be preferred if more figures are required, or if α differs from a tabular value in Table 17.5.

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